EXISTENCE OF BOUND STATES FOR $(N+1)$-COUPLED LONG-WAVE–SHORT-WAVE INTERACTION EQUATIONS

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Abstract. We prove the existence of an infinite family of smooth positive bound states for $(N+1)$-coupled long-wave–short-wave interaction equations. The system describes the interaction between N short waves and a long wave and is of interest in physics and fluid dynamics.

1. Introduction

A non-linear system of interaction between a complex short-wave field u and a real long-wave field v of the form

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv + \beta|u|^2 u \\ \partial_t v + \partial_x^3 v + v\partial_x v = \gamma\partial_x\left(|u|^2\right), \end{cases} \tag{1.1}$$

was first studied in [24] concerning the well-posedness of the Cauchy problem. The system (1.1) which has an interaction between a nonlinear Schrödinger (NLS)-type short wave and a Korteweg de Vries (KdV)-type long wave appears in a wide variety of physical systems. The reader may refer to [3] for a general theory of the non-linear long-wave–short-wave interaction (LSI) model. Numerous successful applications of the LSI model exist in different contexts of fluid dynamics such as capillary-gravity waves in [17], sonic-Langmuir solitons in [16, 25], Alfvén waves in [23], and Bose-Einstein condensates in [20], to mention but a few.

Let us consider the multicomponent LSI system

$$\begin{cases} i\partial_t u_j + \partial_x^2 u_j = \alpha_j u_j v + \beta_j |u_j|^2 u_j, \ j = 1, 2, . . . N, \\ \partial_t v + \partial_x^3 v + v\partial_x v = \dfrac{1}{2}\,\partial_x\left(\displaystyle\sum_{l=1}^{N}\alpha_l|u_l|^2\right), \end{cases} \tag{1.2}$$

where $u_1, ..., u_N$ are complex-valued functions of (x,t), v is a real-valued function of (x,t), and α_j, β_j are real constants. Systems of the form (1.2) arise in the event of a nonlinear interaction between N short waves, namely, $u_1, ..., u_N$, and a long wave v. For more details, readers may see [1, 11,

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16, 17]. The non-linear multicomponent LSI model has not been studied as extensively as its single component counterpart. As such, it has been generating a significant amount of interest in recent years. Some examples of where a multicomponent LSI system may arise are bio-physics [12], ferromagnetism [19], water waves [11], nonlinear optics [21], elastic solid mechanics [15], and acoustics [22].

In this paper, we are concerned with finding bound-state solutions to (1.2). We look for solutions of the form

$$\begin{cases} u_j(x,t) = e^{i\omega t + ikx}\phi_j(x-ct), \ j = 1, ..., N, \\ v(x,t) = \psi(x-ct), \end{cases} \tag{1.3}$$

where ψ and ϕ_j are real-valued functions and $c = 2k$. Inserting (1.3) into (1.2), we find $(\phi_1, \ldots, \phi_N, \psi)$ satisfies the following system of ordinary differential equations:

$$\begin{cases} -\phi_j'' + \sigma\phi_j = -\beta_j\phi_j^3 - \alpha_j\phi_j\psi, \ j = 1, ..., N, \\ -\psi'' + c\psi = \dfrac{\psi^2}{2} - \dfrac{1}{2}\displaystyle\sum_{l=1}^{N}\alpha_l\phi_l^2, \end{cases} \tag{1.4}$$

where $\sigma = k^2 + \omega$, and the primes denote derivatives with respect to the variable $\xi = x - ct$.

To state our main result, let us denote by Y the product space

$$Y = H^1 \times H^1 \times ... \times H^1 \text{ (taken } N+1 \text{ times)},$$

and $\Delta = (u_1, ..., u_N, v)$. For $d > 0$ and $\lambda \geq 0$, we set

$$X_{d,\lambda} = \{\Delta \in Y : d\|v\|_{L^2}^2 + \sum_{j=1}^{N}\|u_j\|_{L^2}^2 = \lambda\} \tag{1.5}$$

and consider the minimization problem

$$I(\lambda) = \inf\{E(\Delta) : \Delta \in X_{d,\lambda}\}, \tag{1.6}$$

where

$$E(\Delta) = \int_{\mathbb{R}}\left(\sum_{j=1}^{N}\left((u_j')^2 + \frac{\beta_j}{2}u_j^4 + \alpha_j v u_j^2\right) + (v')^2 - \frac{1}{3}v^3\right)\, dx \tag{1.7}$$

The following is our main result:

Theorem 1.1. *Suppose that $\alpha_j, \beta_j < 0$ for all $j = 1, ..., N$, and let*

$$\mathcal{M}_\lambda = \{\Delta \in X_{d,\lambda} : I(\lambda) = E(\Delta)\}. \tag{1.8}$$

Then the following results hold:

(i) For every $\lambda > 0$, we have $\mathcal{M}_\lambda \neq \emptyset$.

(ii) If $(\phi_1, ..., \phi_N, \psi) \in \mathcal{M}_\lambda$, then $\psi \geq 0$ and $\phi_j \geq 0$where at least one ϕ_j and ψ are nontrivial. Furthermore, $\psi, \phi_j \in H^\infty$ and decay exponentially at infinity.

(iii) There exists $\lambda^* > 0$ *such that if* $(\phi_1, ..., \phi_N, \psi) \in \mathcal{M}_\lambda$ *for* $\lambda > \lambda^*$, *then* $\psi(x) > 0$ *and* $\phi_j(x) > 0$ *for all* $x \in \mathbb{R}$ *for at least one* $j \in \{1, ..., N\}$.

(iv) There exists a family

$$\begin{cases} u_{j,n}(x,t) = e^{i\omega_n t + ik_n x}\phi_{j,n}(x - c_n t), \ j = 1, ..., N, \\ v_n(x,t) = \psi_n(x - c_n t) \end{cases} \tag{1.9}$$

of non-trivial bound-state solutions to (1.2) *with*

$$\lim_{n\to\infty} c_n = +\infty. \tag{1.10}$$

Moreover, each ψ_n *and* $\phi_{j,n}$ *can be chosen as in parts (ii) and (iii).*

Theorem 1.1 is a generalization of the analogous result proved previously in [13, 14] for (1+1)-coupled system given by (1.1). We also mention the papers [5, 6, 7, 8] where different techniques have been used to prove the existence of solutions for coupled systems of long and short-wave equations. In [2], the existence of a two-parameter family of disjoint sets of bound states has been proved in the case when $N = 1$. We use similar techniques as the ones used in [13, 14], which uses the concentration compactness argument of P.L. Lions [18] and relies on the results of Berestycki and Lions [4], along with an estimate for the Lagrange multiplier of the associated problem.

We remark at this point that the multiconstraint variational problem

$$\inf\{E(\Delta) : \Delta \in Y, \|u_j\|_{L^2}^2 = a_j, \ j = 1, ..., N, \|v\|_{L^2}^2 = a\} \tag{1.11}$$

can also be used to establish the existence of bound-state solutions. The solution set of the problem (1.11) corresponds not only to $(N+1)$-parameter bound-state solutions of (1.2), but also provides a disjoint family of such bound states for each $a_j > 0$ and $a > 0$. However, application of the concentration compactness technique in this situation is complicated by the fact that one needs to establish the strict subadditivity property for the $(N+1)$-parameter problem (1.11) in order to establish the relative compactness of the minimizing sequence. In [9], this problem has been solved under three constraints for the energy functional associated with 3-coupled Schrödinger systems. To our knowledge, the compactness of minimizing sequence for such multiconstraint problems with more than three constraints remains an open question.

2. Existence of Minimizers

We first establish a few lemmas and propositions that will be used in the proof of our main result.

Lemma 2.1. *For all* $\lambda > 0$, *one has* $I(\lambda) > -\infty$.

Proof. Let $\Delta \in X_{d,\lambda}$. Then, upon using the Gagliardo-Nirenberg inequality, we have, for each j,

$$\int u_j^4 \le \|u_j\|_{L^4}^4 \le C_1 \|u_j'\|_{L^2} \|u_j\|_{L^2}^3 \le C_1 \lambda^{3/2} \|u_j'\|_{L^2}$$

and

$$v^3 \le \|v\|_{L^3}^3 \le C_2\|v'\|_{L^2}^{1/2}\|v\|_{L^2}^{5/2} \le C_2\left(\frac{\lambda}{d}\right)^{5/4}\|v'\|_{L^2}^{1/2}.$$

Also, for each j,

$$vu_j^2 \le \frac{1}{2}v^2 + \frac{1}{2}u_j^4 \le \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|u_j\|_{L^4}^4 \le \frac{\lambda}{2d} + \frac{C_1}{2}\lambda^{3/2}\|u_j'\|_{L^2}.$$

Using all of the above estimates, we finally obtain

$$\begin{aligned}
E(\Delta) &= \sum_{j=1}^{N}\left(\|u_j'\|_{L^2}^2 + \frac{\beta_j}{2}\int u_j^4 + \alpha_j\int vu_j^2\right) + \|v'\|_{L^2}^2 - \frac{1}{3}\int v^3 \\
&\ge \sum_{j=1}^{N}\left(\|u_j'\|_{L^2}^2 - \frac{|\beta_j|}{2}\int u_j^4 - |\alpha_j|\int |v|u_j^2\right) + \|v'\|_{L^2}^2 - \frac{1}{3}\int v^3 \\
&\ge \sum_{j=1}^{N}\left(\|u_j'\|_{L^2}^2 - \left(\frac{|\beta_j|}{2}C_1\lambda^{3/2} + \left(|\alpha_j|\left(\frac{\lambda}{2} + \frac{C_1}{2}\lambda^{3/2}\right)\right)\|u_j'\|_{L^2}\right)\right) \\
&\quad + \|v'\|_{L^2}^2 - \frac{C_2}{3}\left(\frac{\lambda}{d}\right)^{5/4}\|v'\|_{L^2}^{1/2}.
\end{aligned}$$

This proves that $E(\Delta)$ is bounded below by an expression depending only on λ and d. ■

Proposition 2.2. *For all $\lambda \ge 0, I(\lambda) \le 0$. Also, there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*, I(\lambda) \le -A\lambda^2$, where A is a positive constant independent of λ.*

Proof. Let $\Delta_1 = (u_1, 0, \dots, 0) \in X_{d,\lambda}$ where $u_1 \in H^1$ is such that $\|u_1\|_{L^2}^2 = \lambda$. Then, since $\beta_1 < 0$, we have

$$E(\Delta_1) = \int\left(u_1'^2 + \frac{\beta_1}{2}u_1^4\right) \le \int u_1'^2.$$

Taking infimum over all $u_1 \in H^1$ such that $\|u_1\|_{L^2}^2 = \lambda$, we get $I(\lambda) \le 0$. Next, let $u_1 \in H^1$ such that $\|u_1\|_{L^2}^2 = 1$ and put $u_{1\lambda} = \lambda^{1/2}u_1$. Then, $\Delta_{1\lambda} = (u_{1\lambda}, 0, \dots, 0) \in X_{d,\lambda}$. Also, using the fact that $\beta_1 < 0$, we can have

$$E(\Delta_{1\lambda}) = \lambda\int u_1'^2 + \frac{\beta_1}{2}\lambda^2\int u_1^4 = \lambda\int u_1'^2 - \frac{|\beta_1|}{4}\lambda^2\int u_1^4 - \frac{|\beta_1|}{4}\lambda^2\int u_1^4.$$

Then, choosing $A = \frac{|\beta_1|}{4}\int u_1^4$ and λ^* in such a way that $\int u_1'^2 - A\lambda^* \le 0$, we will have, for all $\lambda > \lambda^*$,

$$E(\Delta_{1\lambda}) = \lambda\int u_1'^2 - A\lambda^2 - A\lambda^2 \le \lambda A(\lambda^* - \lambda) - A\lambda^2 \le -A\lambda^2.$$

Taking infimum over all $u_1 \in H^1$ such that $\|u_1\|_{L^2}^2 = 1$, we get the result. ■

Lemma 2.3. *For all $\Delta = (f_1, ..., f_N, g) \in Y$, we have*

$$E(|\Delta|) = E(|f_1|, ..., |f_N|, |g|) \le E(f_1, ..., f_N, g) = E(\Delta).$$

Proof. It is immediate from the following two facts: one is that, for $f \in H^1$ real-valued, we have $\||f|'\|_{L^2} \leq \|f'\|_{L^2}$ (for example, see [4]), and the other that $\alpha_j < 0$ yields $\alpha_j u_j^2|v| \leq \alpha_j u_j^2 v$ and $-\frac{1}{3}|v|^3 \leq -\frac{1}{3}v^3$. ■

Lemma 2.4. *I is non-increasing on $[0,\infty)$. That is, for all $\theta > 1$, we have $I(\theta\lambda) \leq \theta I(\lambda)$.*

Proof. Consider a sequence $\{\Delta_k\} = \{(u_{1,k},\dots,u_{N,k},v_k)\} \subset X_{d,\lambda}$ and denote $\sqrt{\theta}\Delta_k = \left(\sqrt{\theta}u_{1,k},\dots,\sqrt{\theta}u_{N,k},\sqrt{\theta}v_k\right)$. Then

$$
\begin{aligned}
&E(\sqrt{\theta}\Delta_k)\\
&= \theta E(\Delta_k) - \frac{(\theta^{3/2}-\theta)}{3}\int v_k^3\\
&\qquad + \sum_{j=1}^{N}\left((\theta^{3/2}-\theta)\alpha_j\int v_k u_{j,k}^2 + \frac{(\theta^2-\theta)}{2}\beta_j\int u_{j,k}^4\right)\\
&\leq \theta E(\Delta_k)\\
&- \min\{(\theta^{3/2}-\theta),(\theta^2-\theta)\}\left[\frac{1}{3}\int v_k^3 + \sum_{j=1}^{N}\left(\frac{|\beta_j|}{2}\int u_{j,k}^4 + |\alpha_j|\int v_k u_{j,k}^2\right)\right]\\
&\leq \theta E(\Delta_k).
\end{aligned}
$$

Next, taking infimum over all sequences Δ_k and using the fact that $\Delta \in X_{d,\lambda}$ if and only if $\sqrt{\theta}\Delta \in X_{d,\theta\lambda}$, we obtain $I(\theta\lambda) \leq \theta I(\lambda)$. ■

Lemma 2.5. *There exists $\lambda_1 \in [0,\lambda^*)$ such that $I(\lambda) < 0 \Leftrightarrow \lambda > \lambda_1$. Furthermore, for all $\lambda > \lambda_1$ and $\theta > 1$, we have $I(\theta\lambda) < \theta I(\lambda)$.*

Proof. By Proposition 2.2 and Lemma 2.4, I is a non-positive and non-increasing function on $[0,\infty)$ which is strictly negative on (λ^*,∞). Hence, there exists $\lambda_1 \in [0,\lambda^*]$ such that I vanishes on $[0,\lambda_1]$ and is strictly negative and decreasing on (λ_1,∞). Then, for any $\lambda > \lambda_1$, we have $I(\lambda) < 0$. Hence, there exists $\delta > 0$ such that we can, as guaranteed by Lemma 2.3, construct a positive minimizing sequence

$$\Delta_k = (u_{1,k},\dots,u_{N,k},v_k) \in X_{d,\lambda}$$

for the minimization problem (1.6) in such a way that $E(\Delta_k) \leq -\delta$ for all $k \in \mathbb{N}\setminus\{0\}$. It exists because otherwise it would give us a subsequence Δ_{k_l} with $E(\Delta_{k_l}) \geq 0$ which, in turn, would lead to a contradiction $I(\lambda) = \lim_{k_l\to\infty} E(\Delta_{k_l}) \geq 0$. So, using the fact that $v_k > 0, \alpha_j < 0$ and $\beta_j < 0$, we get

$$\begin{aligned}
-\delta &\geq E(\Delta_k)\\
&= \sum_{j=1}^{N}\left(\|u'_{j_k}\|_{L^2}^2 + \frac{\beta_j}{2}\int u_{j,k}^4 + \alpha_j \int v_k u_{j,k}^2\right) + \|v'_k\|_{L^2}^2 - \frac{1}{3}\int v_k^3\\
&\geq -\sum_{j=1}^{N}\left(\frac{|\beta_j|}{2}\int u_{j,k}^4 + |\alpha_j|\int v_k u_{j,k}^2\right) - \frac{1}{3}\int v_k^3.
\end{aligned}$$

Combining this result with what we have computed in Lemma 2.4, we obtain

$$\begin{aligned}
&E(\sqrt{\theta}\Delta_k)\\
&\leq \theta E(\Delta_k)\\
&- \min\{(\theta^{3/2}-\theta),(\theta^2-\theta)\}\left[\frac{1}{3}\int v_k^3 + \sum_{j=1}^{N}\left(\frac{|\beta_j|}{2}\int u_{j,k}^4 + |\alpha_j|\int v_k u_{j,k}^2\right)\right]\\
&\leq \theta E(\Delta_k) - \delta\min\{(\theta^{3/2}-\theta),(\theta^2-\theta)\}\\
&< \theta E(\Delta_k).
\end{aligned}$$

The result follows upon taking infimum over all positive minimizing sequences Δ_k and using the fact that $\Delta \in X_{d,\lambda}$ if and only if $\sqrt{\theta}\Delta \in X_{d,\theta\lambda}$. ■

With Lemma 2.5 in hand, an argument very similar to Lemma 2.3 of [21] yields the following strict sub-additivity result for $I(\lambda)$:

Corollary 2.6. *Let $\lambda > \lambda_1$ and $0 < \Omega < \lambda$. Then $I(\lambda) < I(\Omega) + I(\lambda - \Omega)$.*

The following Proposition establishes the existence of minimizers for the minimization problem (1.6):

Proposition 2.7. *For every $\lambda > \lambda_1 \geq 0$, the set $\mathcal{M}_\lambda$ as defined in* (1.8) *is non-empty. Furthermore, if $(\phi_1, \ldots, \phi_N, \psi) \in \mathcal{M}_\lambda$, then the following hold:*

(i) $\psi \geq 0$ and $\phi_j \geq 0$ where ψ and at least one ϕ_j are nontrivial.
(ii) $\psi, \phi_j \in H^\infty$ and decay exponentially at infinity.

Proof. Consider a positive minimizing sequence $\Delta_k = (u_{1,k}, \ldots, u_{N,k}, v_k) \in X_{d,\lambda}$ for the minimization problem (1.6). Set

$$\rho_k = \sum_{j=1}^{N} u_{j,k}^2 + d v_k^2$$

and, following the notations in [18], define the concentration function of ρ_k as

$$Q_k(r) = \sup_{y\in\mathbb{R}}\int_{y-r}^{y+r}\rho_k, \text{ and set } \Omega(r) = \lim_{k\to\infty} Q_k(r) \text{ and } \Omega = \lim_{r\to\infty}\Omega(r).$$

Then there are three possibilities for Ω: vanishing ($\Omega = 0$), dichotomy ($0 < \Omega < \lambda$) and compactness ($\Omega = \lambda$). Our goal is to establish the last alternative. We now divide the proof into three parts.

Part I. The vanishing case does not occur. If $\Omega = 0$, then, by the definition of ρ_k, we have, for each $j = 1, \ldots, N$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} u_{j,k}^2 = \lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} v_k^2 = 0.$$

Using Lemma I.1 of [18] and the fact that $u_{j,k}$ and v_k are bounded in H^1 as seen in the proof of Lemma 2.1, we get

$$\|u_{j,k}\|_{L^p} \to 0, \|v_k\|_{L^p} \to 0, \text{ for all } p > 2.$$

Then, upon using the Cauchy-Schwarz inequality, we have

$$0 \le \lim_{k\to\infty} \int v_k u_{j,k}^2 \le \lim_{k\to\infty} \|v_k\|_{L^2} \|u_{j,k}\|_{L^4}^2 = 0.$$

This implies that

(2.1)

$$\begin{aligned} I(\lambda) &= \lim_{k\to\infty} \inf E(\Delta_k) \\ &= \lim_{k\to\infty} \inf \sum_{j=1}^N \left(\|u'_{j,k}\|_{L^2}^2 - \frac{|\beta_j|}{2} \|u_{j,k}\|_{L^4}^4 - |\alpha_j| \int v_k u_{j,k}^2 \right) + \|v'_k\|_{L^2}^2 - \frac{1}{3} \|v_k^3\|_{L^3}^3 \\ &= \lim_{k\to\infty} \inf \sum_{j=1}^N \|u'_{j,k}\|_{L^2}^2 + \|v'_k\|_{L^2}^2 \ge 0, \end{aligned}$$

which is contradictory to Proposition 2.2, and the vanishing case is ruled out.

Part II. The dichotomy case does not occur. Assume the dichotomy case ($0 < \Omega < \lambda$). Then, using Lemma III.1 of [18], for all $\epsilon > 0$, we can have, for all $k \in \mathbb{N}$,

$$\left| \int \eta_1^2(x) \rho_k(x - y_k) - \Omega \right| < \epsilon \text{ and } \left| \int \eta_2^2(x) \rho_k(x - y_k) - (\lambda - \Omega) \right| < \epsilon,$$

where a sequence $\{y_k\} \subset \mathbb{R}$ and cut-off functions $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$ are so chosen that for some fixed constants $0 < R_1 < \frac{R_2}{2}$,

(i) $0 \le \eta_i \le 1, |\eta'_i| < \epsilon < \epsilon, i = 1, 2$,
(ii) $\eta_1(x) = 1$ for $|x| \le R_1$ and $\eta_1(x) = 0$ for $|x| \ge \frac{R_2}{2}$,
(iii) $\eta_2(x) = 1$ for $|x| \ge R_2$ and $\eta_2(x) = 0$ for $|x| \le \frac{R_2}{2}$,
(iv) $\int_{R_1 \le |x - y_k| \le R_2} \rho_k(x) \le \epsilon$.

Set $u_{j,k}^{(i)} = \eta_i(x - y_k) u_{j,k}$ and $v_k^{(i)} = \eta_i(x - y_k) v_k, i = 1, 2$. Then, using the Sobolev and Cauchy-Schwarz inequalities, the fact that

$$u_{j,k}^2(x) \le \rho(x) \text{ and } dv_k^2(x) \le \rho(x),$$

and the above inequality (iv), we obtain

$$\begin{aligned}
&\int_{R_1\le|x-y_k|\le\frac{R_2}{2}}\left[(u'_{j,k})^2-(u^{(1)'}_{j,k})^2\right]\\
&=\int_{R_1\le|x-y_k|\le\frac{R_2}{2}}\Big[(1-\eta_1^2(x-y_k))(u'_{j,k})^2\\
&\qquad\qquad-\eta_1'^2(x-y_k)u_{j,k}^2-2\eta_1(x-y_k)\eta_1'(x-y_k)u_{j,k}u'_{j,k}\Big]\\
&\ge-\epsilon^3-2C_1\epsilon=-\epsilon^3-C\epsilon.
\end{aligned}$$

The exact same inequality holds when $u^{(2)}_{j,k}$ is used instead of $u^{(1)}_{j,k}$ in the above computation. Hence, we have

$$\begin{aligned}
&\|u'_{j,k}\|^2_{L^2}-\|u^{(1)'}_{j,k}\|^2_{L^2}-\|u^{(2)'}_{j,k}\|^2_{L^2}\\
&=\int_{R_1\le|x-y_k|\le R_2}u'^2_{j,k}-\int_{R_1\le|x-y_k|\le\frac{R_2}{2}}(u^{(1)'}_{j,k})^2-\int_{\frac{R_2}{2}\le|x-y_k|\le R_2}(u^{(2)'}_{j,k})^2\\
&=\int_{R_1\le|x-y_k|\le\frac{R_2}{2}}(u'_{j,k})^2-(u^{(1)'}_{j,k})^2+\int_{\frac{R_2}{2}\le|x-y_k|\le R_2}(u'_{j,k})^2-(u^{(2)'}_{j,k})^2\\
&\ge-2\epsilon^3-2C\epsilon.
\end{aligned}$$

The exact same computation yields

$$\|v'_k\|^2_{L^2}-\|v^{(1)'}_k\|^2_{L^2}-\|v^{(2)'}_k\|^2_{L^2}\ge-2\epsilon^3-2C\epsilon.$$

Furthermore, using the Gagliardo-Nirenberg inequality, we obtain

$$\|u_{j,k}\|^4_{L^4}-\|u^{(1)}_{j,k}\|^4_{L^4}-\|u^{(2)}_{j,k}\|^4_{L^4}\le C\epsilon^3,\ \|v_k\|^3_{L^3}-\|v^{(1)}_k\|^3_{L^3}-\|v^{(2)}_k\|^3_{L^3}\le C\epsilon^{5/2}.$$

Next, Cauchy-Schwarz and Gagliardo-Nirenberg inequalities yield

$$\begin{aligned}
&\int v_k u^2_{j,k}-\int v^{(1)}_k(u^{(1)}_{j,k})^2-\int v^{(2)}_k(u^{(2)}_{j,k})^2\\
&=\int_{R_1\le|x-y_k|\le\frac{R_2}{2}}(1-\eta_1^3(x-y_k))v_k u^2_{j,k}\\
&\qquad+\int_{\frac{R_2}{2}\le|x-y_k|\le R_2}(1-\eta_2^3(x-y_k))v_k u^2_{j,k}\\
&\le\int_{R_1\le|x-y_k|\le R_2}v_k u^2_{j,k}\le\|v_k\|_{L^2}\|u_{j,k}\|^2_{L^4}\le C\epsilon^{5/4}.
\end{aligned}$$

Denoting $\Delta_k^{(i)} = (u_{1,k}^{(i)}, \dots, u_{N,k}^{(i)}, v_k^{(i)}), i = 1, 2,$ and putting together the above computations, we finally obtain

$$\begin{aligned} &E(\Delta_k) \\ &= \sum_{j=1}^{N} \left(\|u'_{j,k}\|_{L^2}^2 - \frac{|\beta_j|}{2} \|u_{j,k}\|_{L^4}^4 - |\alpha_j| \int v_k u_{j,k}^2 \right) + \|v'_k\|_{L^2}^2 - \frac{1}{3} \|v_k^3\|_{L^3}^3 \\ &\geq E(\Delta_k^{(1)}) + E(\Delta_k^{(2)}) - C(\epsilon). \end{aligned}$$

Hence, $I(\lambda) \geq I(\Omega) + I(\lambda - \Omega)$, which contradicts Corollary 2.6.

Part III. The set $\mathcal{M}_\lambda$ is non-empty. Parts I and II imply that the only remaining case is the last one, namely, $\Omega = \lambda$. This implies that the sequence Δ_k is relatively compact up to translations. That is, there exist its subsequence, again denoted $\Delta_k = (u_{1,k}, \dots, u_{N,k}, v_k)$, a sequence $\{y_k\} \subset \mathbb{R}$ and the limit $(\phi_1, \dots, \phi_N, \psi) \in X_{d,\lambda}$ such that

$$\tilde{\Delta}_k = (\tilde{u}_{1,k}, \dots, \tilde{u}_{N,k}, \tilde{v}_k) \rightharpoonup (\phi_1, \dots, \phi_N, \psi) \text{ in } H^1(\mathbb{R}),$$

where $\tilde{u}_{j,k} = u_{j,k}(\cdot - y_k), j = 1, \dots, N$ and $\tilde{v}_k = v_k(\cdot - y_k)$. Consequently, we have, for all $p \geq 2$,

$$\tilde{\Delta}_k \to (\phi_1, \dots, \phi_N, \psi) \text{ in } L^p(\mathbb{R}) \text{ and } \int v_k u_{j,k}^2 \to \int \psi \phi^2. \tag{2.2}$$

Computing as in (2.1) and using (2.2), we obtain

$$I(\lambda) = \liminf_{k \to \infty} E(\tilde{\Delta}_k) = E((\phi_1, \dots, \phi_N, \psi)). \tag{2.3}$$

But, on the other hand, we have,

$$E((\phi_1, \dots, \phi_N, \psi)) \geq \inf_{\Delta \in X_{d,\lambda}} E(\Delta) = I(\lambda).$$

This results in $E((\phi_1, \dots, \phi_N, \psi)) = I(\lambda)$ and we thus have $(\phi_1, \dots, \phi_N, \psi) \in \mathcal{M}_\lambda \neq \emptyset$. The computation in (2.3) also establishes

$$\liminf_{k \to \infty} \sum_{j=1}^{N} \|\tilde{u}'_{j,k}\|_{L^2}^2 + \|\tilde{v}'_k\|_{L^2}^2 = \sum_{j=1}^{N} \|\phi'_j\|_{L^2}^2 + \|\psi'\|_{L^2}^2,$$

which leads to

$$\tilde{\Delta}_k = (\tilde{u}_{1,k}, \dots, \tilde{u}_{N,k}, \tilde{v}_k) \to (\phi_1, \dots, \phi_N, \psi) \text{ in } H^1(\mathbb{R}).$$

Then, since $\tilde{u}_{j,k}, \tilde{v}_k \geq 0$, we must have $\phi_j, \psi \geq 0$. Note that $(\phi_1, \dots, \phi_N, \psi) \in X_{d,\lambda}$ if and only if $(\phi_1, \dots, \phi_N, \psi + a) \in X_{d,\lambda}$, for any $a > 0$. Furthermore, due to the terms associated to ψ being negative, we have

$$E((\phi_1, \dots, \phi_N, \psi + a)) < E((\phi_1, \dots, \phi_N, \psi)),$$

which, if $\psi = 0$, would be contradictory to the fact that

$$E((\phi_1, \dots, \phi_N, \psi)) = I(\lambda) = \inf\{E(\Delta) : \Delta \in X_{d,\lambda}\},$$

and hence, $\psi \not\equiv 0$ on $\mathbb{R}$. Using the same argument as in the proof of Proposition 4.5 of [13], one can see that $\phi_j \not\equiv 0$ for at least one $j \in \{1, ..., N\}$.

The smoothness of $\phi_j, j = 1, \ldots, N$ and ψ can be established by a standard bootstrap technique. Next, choosing $c = 2k > 0$ and $\sigma = k^2 + \omega > 0$, and using Theorem 8.1.1 in [10], there exists a $\varepsilon > 0$ such that $e^{\varepsilon|\cdot|}\phi_j, e^{\varepsilon|\cdot|}\psi \in L^\infty$. Hence the functions $\phi_j, j = 1, \ldots, N$ and ψ are in H^∞ and decay exponentially at infinity. ■

3. Proof of the main theorem

In this section, we establish our main result. Parts (i) and (ii) have already been established. To prove the remaining statements, we first prove two ancillary results in the form of a lemma and a proposition.

Lemma 3.1. *There exists a constant $A > 0$ and $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, the Lagrange multiplier μ satisfies*

$$\mu \leq -A\lambda.$$

Proof. By Proposition 2.7, there exists $(\phi_1, \ldots, \phi_N, \psi) \in \mathcal{M}_\lambda$ such that $\psi > 0$ and $\phi_j \geq 0, j = 1, \ldots, N$. There exists a Lagrange multiplier μ depending only on λ such that

$$\begin{cases} -\phi_j'' - \mu\phi_j = -\beta_j\phi_j^3 - \alpha_j\phi_j\psi, \ j = 1, ..., N, \\ -\psi'' - \mu d\psi = \dfrac{\psi^2}{2} - \dfrac{1}{2}\displaystyle\sum_{l=1}^N \alpha_l\phi_l^2. \end{cases}$$

Note that there are $N + 1$ equations in total, keeping in mind that one or more of them, determined by how many of $\phi_j, j = 1, \ldots, N$ are trivial, might vanish. Multiplying each of the $N + 1$ equations by $\phi_j, j = 1, \ldots, N$ and ψ respectively, and integrating by part,

$$\begin{cases} \displaystyle\int \phi_j'^2 - \mu\int \phi_j^2 = -\beta_j\int \phi_j^4 - \alpha_j\int \phi_j^2\psi, \ j = 1, ..., N, \\ \displaystyle\int \psi'^2 - \mu d\int \psi^2 = \frac{1}{2}\int \psi^3 - \frac{1}{2}\int \psi\sum_{l=1}^N \alpha_l\phi_l^2. \end{cases}$$

Adding,

$$\begin{aligned} &\sum_{j=1}^N \|\phi_j'\|_{L^2}^2 + \|\psi'\|_{L^2}^2 - \mu\lambda \\ &= \frac{1}{2}\|\psi\|_{L^3}^3 - \int \left[\sum_{j=1}^N \left(\alpha_j\phi_j^2\psi + \beta_j\phi_j^4 + \frac{\psi}{2}\sum_{l=1}^N \alpha_l\phi_l^2\right)\right]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &I(\lambda) = E((\phi_1, \ldots, \phi_N, \psi)) = \\ &\sum_{j=1}^N \left(\|\phi_j'\|_{L^2}^2 + \frac{\beta_j}{2}\|\phi_j\|_{L^4}^4 + \alpha_j\int \psi\phi_j^2\right) + \|\psi'\|_{L^2}^2 - \frac{1}{3}\|\psi\|_{L^3}^3 \end{aligned}$$

Hence,

$$\begin{aligned}\mu\lambda &= I(\lambda) + \sum_{j=1}^{N} \frac{1}{2}\int \sum_{l=1}^{N} \alpha_l \phi_l^2 \psi + \frac{\beta_j}{2}\|\phi_j\|_{L^4}^4 - \frac{1}{6}\|\psi\|_{L^3}^3 \\ &\leq I(\lambda) \leq -A\lambda^2,\end{aligned}$$

for some $A > 0$ independent of λ, and for all $\lambda > \lambda^*$, as given by Proposition 2.2. ■

Proposition 3.2. *There exists $\lambda^* > 0$ such that if $(\phi_1, ...\phi_N, \psi) \in \mathcal{M}_\lambda$ for $\lambda > \lambda^*$, then we have $\psi(x) > 0$ and $\phi_j(x) > 0$ for all $x \in \mathbb{R}$ for at least one $j \in \{1, ..., N\}$.*

Proof. Let A and λ^* be as in Lemma 3.1. Then $s = -\mu > 0$. For each $j = 1, ..., N$, the functions ψ and ϕ_j satisfy

$$\begin{cases} -\phi_j'' + s\phi_j = -\beta_j \phi_j^3 - \alpha_j \phi_j \psi, \ j = 1, 2, ..., N, \\ -\psi'' + sd\psi = \dfrac{\psi^2}{2} - \dfrac{1}{2}\displaystyle\sum_{l=1}^{N} \alpha_l \phi_l^2. \end{cases} \tag{3.1}$$

Let $P_s(x) = \frac{1}{2\sqrt{s}} e^{-\sqrt{s}|x|}$ for $x \in \mathbb{R}$. Then (3.1) can be written as

$$\begin{cases} \phi_j = P_s \star \left(b_j \phi_j^3 + a_j \phi_j \psi\right), \ j = 1, 2, ..., N, \\ \psi = P_{sd} \star \left(\dfrac{\psi^2}{2} + \dfrac{1}{2}\displaystyle\sum_{l=1}^{N} a_l \phi_l^2\right). \end{cases}$$

where $b_j = -\beta_j > 0$ and $a_j = -\alpha_j > 0$. Since the convolution of the positive kernel P_s with a nonnegative and not identically zero function always produces a positive function, it follows that ϕ_j is positive on all of $\mathbb{R}$ for at least one $j \in \{1, ..., N\}$. ■

We now complete the proof of our main result.

Proof of Theorem 1.1: Parts (i) and (ii) have been established in Proposition 2.7 while part (iii) has been proved in Proposition 3.2. Next, we establish the remaining part (iv). By Lemma 3.1, we have

$$\mu \leq -A\lambda < 0.$$

Now, choose a sequence $\lambda_n \to \infty$. For each n, there will be a Lagrange multiplier μ_n (depending only on λ_n) such that $\mu_n \to -\infty$. Corresponding to each Lagrange multiplier μ_n are solutions $\phi_{j,n}, j = 1, \ldots, N$ and ψ_n. Set

$$c_n = -\mu_n, \ k_n = -\frac{1}{2}\mu_n, \ \omega_n = -\mu_n - k_n^2, \ v_n(x,t) = \psi_n(x - c_n t)$$

and

$$u_{j,n}(x,t) = e^{i\omega_{j,n}t + ik_n x} \phi_{j,n}(x - c_n t), \ j = 1, ..., N.$$

Then, indeed,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} -\mu_n = +\infty.$$

Furthermore, since $k_n = -\frac{1}{2}\mu_n > 0$ and $\sigma_n = k_n^2 + \omega_n = -\mu_n > 0$, a similar argument as in the proof of Proposition 2.7 establishes that the functions $\phi_{j,n}, j = 1, \ldots, N$ and ψ_n are in H^∞ and decay exponentially at infinity.

■

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